VNU Journal of Science: Comp. Science & Com. Eng., Vol. 32, No. 3 (2016) 39–52

Multi-criteria Group Decision Making with Picture Linguistic Numbers

Pham Hong Phong[1,*], Bui Cong Cuong[2]

[1] *Faculty of Information Technology, National University of Civil Engineering, 55 Giai Phong Road, Hanoi, Vietnam*

[2] *Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet Road, Building A5, Cau Giay, Hanoi, Vietnam*

Abstract

In 2013, Cuong and Kreinovich defined picture fuzzy set (PFS) which is a direct extension of fuzzy set (FS) and intuitionistic fuzzy set (IFS). Wang et al. (2014) proposed intuitionistic linguistic number (ILN) as a combination of IFS and linguistic approach. Motivated by PFS and linguistic approach, this paper introduces the concept of picture linguistic number (PLN), which constitutes a generalization of ILN for picture circumstances. For multi-criteria group decision making (MCGDM) problems with picture linguistic information, we define a score index and two accuracy indexes of PLNs, and propose an approach to the comparison between two PLNs. Simultaneously, some operation laws for PLNs are defined and the related properties are studied. Further, some aggregation operations are developed: picture linguistic arithmetic averaging (PLAA), picture linguistic weighted arithmetic averaging (PLWAA), picture linguistic ordered weighted averaging (PLOWA) and picture linguistic hybrid averaging (PLHA) operators. Finally, based on the PLWAA and PLHA operators, we propose an approach to handle MCGDM under PLN environment.

Received 18 March 2016, Revised 07 October 2016, Accepted 18 October 2016

Keywords: Picture fuzzy set, linguistic aggregation operator, multi-criteria group decision making, linguistic group decision making.

1. Introduction

Cuong and Kreinovich [7] introduced the concept of picture fuzzy set (PFS), which is a generalization of the traditional fuzzy set (FS) and the intuitionistic fuzzy set (IFS). Basically, a PFS assigns to each element a positive degree, a neural degree and a negative degree. PFS can be applied to situations that require human opinions involving answers of types: "yes", "abstain", "no" and "refusal". Voting can be a good example of such situation as the voters may be divided into four groups: "vote for", "abstain", "vote against" and "refusal of voting". There has been a number of studies that show the applicability of PFSs (for example, see [18, 19, 20]).

Moreover, in many decision situations, experts' preferences or evaluations are given by linguistic terms which are linguistic values

* Corresponding author. Email.: phphong84@yahoo.com

of a linguistic variable [32]. For example, when evaluating a cars speed, linguistic terms like "very fast", "fast" and "slow" can be used. To date, there are many methods proposed to dealing with linguistic information. These methods are mainly divided into three groups.
1) The methods based on membership functions: each linguistic term is represented as a fuzzy number characterized by a membership function. These methods compute directly on the membership functions using the Extension Principle [13]. Herrera and Martínez [11] described an aggregation operator based on membership functions by

$$S^n \xrightarrow{\tilde{F}} \mathcal{F}(\mathbb{R}) \xrightarrow{\text{app}_1} S,$$

where S^n denotes the n-Cartesian product of the linguistic term set S, $\tilde{F}$ symbolizes an aggregation operator, $\mathcal{F}(\mathbb{R})$ denotes the set of fuzzy numbers, and app_1 is an approximation function that returns a linguistic term in S whose meaning is the closest one to each obtained unlabeled fuzzy number in $\mathcal{F}(\mathbb{R})$. In some early applications, linguistic terms were described via triangular fuzzy numbers [1, 4, 15], or trapezoidal fuzzy numbers [5, 14].
2) The methods based on ordinal scales: the main idea of this approach is to consider the linguistic terms as ordinal information [28]. It is assumed that there is a linear ordering on the linguistic term set $S = \left\{s_0, s_1, \ldots, s_g\right\}$ such that $s_i \geq s_j$ if and only if $i \geq j$.

Based on elementary notions: maximum, minimum and negation, many aggregation operators have been proposed [9, 10, 12, 21, 24, 29, 30].

In 2008, Xu [24] introduced a computational model to improve the accuracy of linguistic aggregation operators by extending the linguistic term set, $S = \left\{s_0, s_1, \ldots, s_g\right\}$, to the continuous one, $\bar{S} = \{s_\theta | \theta \in [0, t]\}$, where t $(t > g)$ is a sufficiently large positive integer. For $s_\theta \in \bar{S}$, if $s_\theta \in S$, s_θ is called an original linguistic term; otherwise, an extended (or virtual) linguistic term. Based on this representation, some aggregation operators were defined: linguistic averaging (LA) [26], linguistic weighted averaging (LWA) [26], linguistic ordered weighted averaging (LOWA) [26], linguistic hybrid aggregation (LHA) [27], induced LOWA (ILOWA) [26], generalized ILOWA (GILOWA) [25] operators.
3) The methods based on 2-tuple representation: Herrera and Martínez [11] proposed a new linguistic computational model using an added parameter to each linguistic term. This new parameter is called sybolic translation. So, linguistic information is presented as a 2-tuple (s, α), where s is a linguistic term, and α is a numeric value representing a sybolic translation. This model makes processes of computing with linguistic terms easily without loss of information. Some aggregation operation for 2-tuple representation were also defined [11]: 2-tuple arithmetic mean (TAM), 2-tuple weighted averaging (TWA), 2-tuple ordered weighted averaging (TOWA) operators.

Motivated by Atanassov's IFSs [2, 3], Wang et al. [22, 23] proposed intuitionistic linguistic number (ILN) as a relevant tool to modelize decision situations in which each assessment consists of not only a linguistic term but also a membership degree and a nonmembership degree. Wang also defined some operation laws and aggregation for ILNs: intuitionistic linguistic arithmetic averaging [22] (ILAA), intuitionistic linguistic weighted arithmetic averaging (ILWAA) [22], intuitionistic linguistic ordered weighted averaging (ILOWA) [23] and intuitionistic hybrid aggregation [23] (IHA) operators. Another concept, which also generalizes both the linguistic term and the intuitionistic fuzzy value at the same time, is intuitionistic linguistic term [6, 8, 16, 17].

The rest of the paper is organized as follows. Section 2 recalls some relevant definitions: picture fuzzy sets and intuitionistic fuzzy numbers. Section 3 introduces the concept of picture linguistic number (PLN), which is a generalization of ILN for picture circumstances. In Section 4, some aggregation operations are developed: picture linguistic arithmetic averaging (PLAA), picture linguistic weighted arithmetic averaging (PLWAA), picture linguistic ordered weighted averaging (PLOWA) and picture linguistic hybrid averaging (PLHA) operators. In Section 5, based on the PLWAA and PLHA operators, we propose an approach to handle MCGDM under PLNs environment. Section 6 is an illutrative example of the proposed approach. Finally, Section 7 draws a conclusion.

2. Related works

2.1. Picture fuzzy sets

Definition 1. *[7] A picture fuzzy set (PFS) A in a set $X \neq \emptyset$ is an object of the form*

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \,|\, x \in X\}, \quad (1)$$

where μ_A, η_A, $\nu_A : X \to [0, 1]$. *For each* $x \in X$, $\mu_A(x)$, $\eta_A(x)$ *and* $\nu_A(x)$ *are correspondingly called the positive degree, neutral degree and negative degree of x in A, which satisfy*

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1, \forall x \in X. \quad (2)$$

For each $x \in X$, $\xi_A(x) = 1-\mu_A(x)-\eta_A(x)-\nu_A(x)$ is termed as the refusal degree of x in A. If $\xi_A(x) = 0$ for all $x \in X$, A is reduced to an IFS [2, 3]; and if $\eta_A(x) = \xi_A(x) = 0$ for all $x \in X$, A is degenerated to a FS [31].

Example 1. *Let A denotes the set of all patients who suffer from "high blood pressure". We assume that, assessments of* 20 *physicians on blood pressure of the patient x are divided into four groups: "high blood pressure" (*7 *physicians), "low blood pressure" (*4 *physicians), "blood pressure disease" (*3 *physicians), " not blood disease pressure" (*6 *physicians). The set A can be considered as a PFS. The possitive degree, neural degree, negative degree and refusal degree of the patient x in A can be specified as follows.*

$$\mu_A(x) = \frac{7}{20} = 0.35, \quad \eta_A(x) = \frac{3}{20} = 0.15,$$
$$\nu_A(x) = \frac{4}{20} = 0.2, \quad \xi_A(x) = 0.3.$$

Some more definitions, properties of PFSs can be referred to [7].

2.2. Intuitionistic linguistic numbers

From now on, the continuous linguistic term set $\bar{S} = \{s_\theta | \theta \in [0, t]\}$ is used as linguistic scale for linguistic assessments.

Let $X \neq \emptyset$, based on the linguistic term set and the intuitionistic fuzzy set [2, 3], Wang and Li [22] defined the intuitionistic linguistic number set as follows.

$$A = \left\{ (x, \langle s_{\theta(x)}, \mu_A(x), \nu_A(x) \rangle) \middle| \, x \in X \right\}, \quad (3)$$

which is characterized by a linguistic term $s_{\theta(x)}$, a membership degree $\mu_A(x)$ and a non-membership degree $\nu_A(x)$ of the element x to $s_\theta(x)$, where

$$\mu_A : X \to \bar{S} \to [0, 1], x \mapsto s_{\theta(x)} \mapsto \mu_A(x), \quad (4)$$

$$\nu_A : X \to \bar{S} \to [0,1], x \mapsto s_{\theta(x)} \mapsto \nu_A(x), \quad (5)$$

with the condition

$$\mu_A(x) + \nu_A(x) \le 1, \forall x \in X. \quad (6)$$

Each $\langle s_{\theta(x)}, \mu_A(x), \nu_A(x)\rangle$ defined in (3) is termed as an intuitionistic linguistic number which exactly given in Definition 2.

Definition 2. *[22] An intuitionistic linguistic number (ILN) α is defined as $\alpha = \langle s_{\theta(\alpha)}, \mu(\alpha), \nu(\alpha)\rangle$, where $s_{\theta(\alpha)} \in \bar{S}$ is a linguistic term, $\mu(\alpha) \in [0,1]$ (resp. $\nu(\alpha) \in [0,1]$) is the membership degree (resp. non-membership degree) such that $\mu(\alpha) + \nu(\alpha) \le 1$. The set of all ILNs is denoted by Ω.*

Definition 3. *[22] Let $\alpha, \beta \in \Omega$, then*
(1) $\alpha \oplus \beta = \left\langle s_{\theta(\alpha)+\theta(\beta)}, \frac{\theta(\alpha)\mu(\alpha)+\theta(\beta)\mu(\beta)}{\theta(\alpha)+\theta(\beta)}, \frac{\theta(\alpha)\nu(\alpha)+\theta(\beta)\nu(\beta)}{\theta(\alpha)+\theta(\beta)}\right\rangle$;
(2) $\lambda\alpha = \langle s_{\lambda\theta(\alpha)}, \mu(\alpha), \nu(\alpha)\rangle$, for all $\lambda \in [0,1]$.

Definition 4. *[23] For $\alpha \in \Omega$, the score $h(\alpha)$ and the accuracy $H(\alpha)$ of α are respectively given in Eqs. (7) and (8).*

$$h(\alpha) = \theta(\alpha)(\mu(\alpha) - \nu(\alpha)), \quad (7)$$

$$H(\alpha) = \theta(\alpha)(\mu(\alpha) + \nu(\alpha)). \quad (8)$$

Definition 5. *[23] Consider $\alpha, \beta \in \Omega$, α is said to be greater than β, denoted by $\alpha > \beta$, if one of the following conditions is satisfied.*
(1) If $h(\alpha) > h(\beta)$;
(2) If $h(\alpha) = h(\beta)$, and $H(\alpha) > H(\beta)$.

Based on basic operators (Definition 3) and order relation (Definition 5), Wang et al. defined the intuitionistic linguistic weighted arithmetic averaging [22], intuitionistic linguistic ordered weighted averaging [23], intuitionistic linguistic hybrid aggregation operator [23] operators, and developed an approach to deal with the MCGDM problems, in which the criteria values are ILNs [23] .

3. Picture linguistic numbers

Definition 6. *Let $X \neq \emptyset$, then a picture linguistic number set A in X is an object having the following form:*

$$A = \left\{ (x, \langle s_{\theta(x)}, \mu_A(x), \eta_A(x), \nu_A(x)\rangle) \middle| x \in X \right\}, \quad (9)$$

which is characterized by a linguistic term $s_{\theta(x)} \in \bar{S}$, a positive degree $\mu_A(x) \in [0,1]$, a neural degree $\eta_A(x) \in [0,1]$ and a negative degree $\nu_A(x) \in [0,1]$ of the element x to $s_{\theta(x)}$ with the condition

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \le 1, \forall x \in X. \quad (10)$$

$\xi_A(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x)$ is called the refusal degree of x to $s_{\theta(x)}$ for all $x \in X$.

In cases $\eta_A(x) = 0$ (for all $x \in X$), the picture linguistic number set is returns to the intuitionistic linguistic number set [22].

For convenience, each 4-tuple $\alpha = \langle s_{\theta(\alpha)}, \mu(\alpha), \eta(\alpha), \nu(\alpha)\rangle$ is called a picture linguistic number (PLN), where $s_{\theta(\alpha)}$ is a linguistic term, $\mu(\alpha) \in [0,1]$, $\eta(\alpha) \in [0,1]$, $\nu(\alpha) \in [0,1]$ and $\mu(\alpha) + \eta(\alpha) + \nu(\alpha) \le [0,1]$. $\mu(\alpha)$, $\eta(\alpha)$ and $\nu(\alpha)$ are membership, neutral and nonmembership degrees of an evaluated object to $s_{\theta(\alpha)}$, respectively. Two PLNs α and β are said to be equal, $\alpha = \beta$, if $\theta(\alpha) = \theta(\alpha)$, $\mu(\alpha) = \mu(\beta)$, $\eta(\alpha) = \eta(\beta)$ and $\nu(\alpha) = \nu(\beta)$. Let Δ denotes the set of all PLNs.

Example 2. *$\alpha = \langle s_4, 0.3, 0.3, 0.2\rangle$ is a PLN, and from it, we know that the positive* *degree, neural degree, negative degree and the refusal degree of evaluated object to* s_4 *are* 0.3, 0.3, 0.2 *and* 0.2, *respectively.*

In the following, some operational laws of PLNs are introduced.

Definition 7. *Let* $\alpha, \beta \in \Delta$, *then*
(1) $\alpha \oplus \beta = \left\langle s_{\theta(\alpha)+\theta(\beta)}, \frac{\theta(\alpha)\mu(\alpha)+\theta(\beta)\mu(\beta)}{\theta(\alpha)+\theta(\beta)}, \frac{\theta(\alpha)\eta(\alpha)+\theta(\beta)\eta(\beta)}{\theta(\alpha)+\theta(\beta)}, \frac{\theta(\alpha)\nu(\alpha)+\theta(\beta)\nu(\beta)}{\theta(\alpha)+\theta(\beta)} \right\rangle$;
(2) $\lambda\alpha = \langle s_{\lambda\theta(\alpha)}, \mu(\alpha), \eta(\alpha), \nu(\alpha)\rangle$, *for all* $\lambda \in [0, 1]$.

It is easy to prove that both $\alpha \oplus \beta$ and $\lambda\alpha$ ($\lambda \in [0, 1]$) are PLNs. Proposition 1 further examines properties of aforesaid notions.

Proposition 1. *Let* α, β, $\gamma \in \Delta$, *and* $\lambda, \rho \in [0, 1]$, *we have:*
(1) $\alpha \oplus \beta = \beta \oplus \alpha$;
(2) $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$;
(3) $\lambda(\alpha \oplus \beta) = \lambda\alpha \oplus \lambda\beta$;
(4) If $\lambda + \rho \leq 1$, $(\lambda + \rho)\alpha = \lambda\alpha \oplus \rho\alpha$.

Proof. (1) It is straightforward.
(2) We have

$$\theta((\alpha \oplus \beta) \oplus \gamma) = \theta(\alpha) \oplus \theta(\beta) \oplus \theta(\gamma).$$

$$\begin{aligned} &\mu((\alpha \oplus \beta) \oplus \gamma) \\ &= \left((\theta(\alpha) + \theta(\beta)) \frac{\theta(\alpha)\eta(\alpha) + \theta(\beta)\eta(\beta)}{\theta(\alpha) + \theta(\beta)}\right. \\ &+ \theta(\gamma)\mu(\gamma)) / (\theta(\alpha) + \theta(\beta) + \theta(\gamma)) \\ &= \frac{\theta(\alpha)\mu(\alpha) + \theta(\beta)\mu(\beta) + \theta(\gamma)\mu(\gamma)}{\theta(\alpha) + \theta(\beta) + \theta(\gamma)}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\eta((\alpha \oplus \beta) \oplus \gamma) \\ &= \frac{\theta(\alpha)\eta(\alpha) + \theta(\beta)\eta(\beta) + \theta(\gamma)\eta(\gamma)}{\theta(\alpha) + \theta(\beta) + \theta(\gamma)}, \end{aligned}$$

and

$$\begin{aligned} &\nu((\alpha \oplus \beta) \oplus \gamma) \\ &= \frac{\theta(\alpha)\nu(\alpha) + \theta(\beta)\nu(\beta) + \theta(\gamma)\nu(\gamma)}{\theta(\alpha) + \theta(\beta) + \theta(\gamma)}. \end{aligned}$$

Hence,

$$\begin{aligned} (\alpha \oplus \beta) \oplus \gamma = \langle &\theta(\alpha) \oplus \theta(\beta) \oplus \theta(\gamma), \\ &\frac{\theta(\alpha)\mu(\alpha) + \theta(\beta)\mu(\beta) + \theta(\gamma)\mu(\gamma)}{\theta(\alpha) + \theta(\beta) + \theta(\gamma)} \\ &\frac{\theta(\alpha)\eta(\alpha) + \theta(\beta)\eta(\beta) + \theta(\gamma)\eta(\gamma)}{\theta(\alpha) + \theta(\beta) + \theta(\gamma)}, \\ &\left.\frac{\theta(\alpha)\nu(\alpha) + \theta(\beta)\nu(\beta) + \theta(\gamma)\nu(\gamma)}{\theta(\alpha) + \theta(\beta) + \theta(\gamma)}\right\rangle. \end{aligned} \tag{11}$$

By the same way, $\alpha \oplus (\beta \oplus \gamma)$ equals to the right of Eq. (11). Therefore, $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$.
(3) We have

$$\begin{aligned} &\lambda(\alpha \oplus \beta) = \left\langle s_{\lambda(\theta(\alpha)+\theta(\beta))}, \right. \\ &\frac{\theta(\alpha)\mu(\alpha) + \theta(\beta)\mu(\beta)}{\theta(\alpha) + \theta(\beta)}, \frac{\theta(\alpha)\eta(\alpha) + \theta(\beta)\eta(\beta)}{\theta(\alpha) + \theta(\beta)}, \\ &\left.\frac{\theta(\alpha)\nu(\alpha) + \theta(\beta)\nu(\beta)}{\theta(\alpha) + \theta(\beta)}\right\rangle \\ &= \left\langle s_{\lambda\theta(\alpha)+\lambda\theta(\beta)}, \frac{\lambda\theta(\alpha)\mu(\alpha) + \lambda\theta(\beta)\mu(\beta)}{\lambda\theta(\alpha) + \lambda\theta(\beta)}, \right. \\ &\frac{\lambda\theta(\alpha)\eta(\alpha) + \lambda\theta(\beta)\eta(\beta)}{\lambda\theta(\alpha) + \lambda\theta(\beta)}, \\ &\left.\frac{\lambda\theta(\alpha)\nu(\alpha) + \lambda\theta(\beta)\nu(\beta)}{\lambda\theta(\alpha) + \lambda\theta(\beta)}\right\rangle \\ &= \langle s_{\lambda\theta(\alpha)}, \mu(\alpha), \eta(\alpha), \nu(\alpha)\rangle \\ &\oplus \left\langle s_{\lambda\theta(\beta)}, \mu(\beta), \eta(\beta), \nu(\beta)\right\rangle \\ &= \lambda\alpha \oplus \lambda\beta. \end{aligned}$$

(4) We have

$$\begin{aligned} &(\lambda + \rho)\alpha = \left\langle s_{(\lambda+\rho)\theta(\alpha)}, \mu(\alpha), \eta(\alpha), \nu(\alpha)\right\rangle \\ &= \left\langle s_{\lambda\theta(\alpha)+\rho\theta(\alpha)}, \frac{\lambda\theta(\alpha)\mu(\alpha) + \rho\theta(\alpha)\mu(\alpha)}{\lambda\theta(\alpha) + \rho\theta(\alpha)}, \right. \\ &\frac{\lambda\theta(\alpha)\eta(\alpha) + \rho\theta(\alpha)\eta(\alpha)}{\lambda\theta(\alpha) + \rho\theta(\alpha)}, \\ &\left.\frac{\lambda\theta(\alpha)\nu(\alpha) + \rho\theta(\alpha)\nu(\alpha)}{\lambda\theta(\alpha) + \rho\theta(\alpha)}\right\rangle \\ &= \langle s_{\lambda\theta(\alpha)}, \mu(\alpha), \eta(\alpha), \nu(\alpha)\rangle \\ &\oplus \left\langle s_{\rho\theta(\alpha)}, \mu(\alpha), \eta(\alpha), \nu(\alpha)\right\rangle \\ &= \lambda\alpha \oplus \rho\alpha. \end{aligned}$$

In order to compare two PLNs, we define the score, first accuracy and second accuracy for PLNs.

Definition 8. *We define the score $h(\alpha)$, first accuracy $H_1(\alpha)$ and second accuracy $H_2(\alpha)$ for $\alpha \in \Delta$ as in Eqs. (12), (13) and (14).*

$$h(\alpha) = \theta(\alpha)\left(\mu(\alpha) - \nu(\alpha)\right), \tag{12}$$

$$H_1(\alpha) = \theta(\alpha)\left(\mu(\alpha) + \nu(\alpha)\right), \tag{13}$$

$$H_2(\alpha) = \theta(\alpha)\left(\mu(\alpha) + \eta(\alpha) + \nu(\alpha)\right). \tag{14}$$

Definition 9. *For $\alpha, \beta \in \Delta$, α is said to be greater than β, denoted by $\alpha > \beta$, if one of following three cases is satisfied:*
(1) $h(\alpha) > h(\beta)$;
(2) $h(\alpha) = h(\beta)$ and $H_1(\alpha) > H_1(\beta)$;
(3) $h(\alpha) = h(\beta)$, $H_1(\alpha) = H_1(\beta)$ and $H_2(\alpha) > H_2(\beta)$.

It is easy seen that there exist pairs of PLNs which are not comparable by Definition 9. For example, let us consider $\alpha = \langle s_2, 0.4, 0.2, 0.2\rangle$ and $\beta = \langle s_4, 0.2, 0.1, 0.1\rangle$. We have $h(\alpha) = h(\beta)$, $H_1(\alpha) = H_1(\beta)$ and $H_2(\alpha) = H_2(\beta)$. Then, neither $\alpha \geq \beta$ nor $\beta \geq \alpha$ occurs. In these cases, α and β are said to be equivalent.

Definition 10. *Two PLNs α and β are termed as equivalent, denoted by $\alpha \sim \beta$, if they have the same score, first accuracy and second accuracy, that is $h(\alpha) = h(\beta)$, $H_1(\alpha) = H_1(\beta)$ and $H_2(\alpha) = H_2(\beta)$.*

Proposition 2. *Let us consider $\alpha, \beta, \gamma \in \Delta$, then*
(1) There are only three cases of the relation between α and β: $\alpha > \beta$, $\beta > \alpha$ or $\alpha \sim \beta$.
(2) If $\alpha > \beta$ and $\beta > \gamma$, then $\alpha > \gamma$;

Proof. (1) We assume that $\alpha \not> \beta$ and $\beta \not> \alpha$. By Definition 9,

$$\alpha \not> \beta \Leftrightarrow$$
$$\begin{cases} h(\alpha) \leq h(\beta) \\ h(\alpha) \neq h(\beta) \text{ or } H_1(\alpha) \leq H_1(\beta) \\ h(\alpha) \neq h(\beta) \text{ or } H_1(\alpha) \neq H_1(\beta) \\ \qquad \text{or } H_2(\alpha) \leq H_2(\beta), \end{cases} \tag{15}$$

and

$$\beta \not> \alpha \Leftrightarrow$$
$$\begin{cases} h(\beta) \leq h(\alpha) \\ h(\beta) \neq h(\alpha) \text{ or } H_1(\beta) \leq H_1(\alpha) \\ h(\beta) \neq h(\alpha) \text{ or } H_1(\beta) \neq H_1(\alpha) \\ \qquad \text{or } H_2(\beta) \leq H_2(\alpha). \end{cases} \tag{16}$$

Combining (15) and (16), we get $h(\alpha) = h(\beta)$, $H_1(\alpha) = H_1(\beta)$ and $H_2(\alpha) = H_2(\beta)$. Thus $\alpha \sim \beta$.
(2) Taking account of Definition 9, we get

$$\left[\begin{array}{l} h(\alpha) > h(\beta) \\ h(\alpha) = h(\beta) \text{ and } H_1(\alpha) > H_1(\beta) \\ h(\alpha) = h(\beta) \text{ and } H_1(\alpha) = H_1(\beta) \\ \qquad \text{and } H_2(\alpha) > H_2(\beta), \end{array}\right. \tag{17}$$

and

$$\left[\begin{array}{l} h(\beta) > h(\gamma) \\ h(\beta) = h(\gamma) \text{ and } H_1(\beta) > H_1(\gamma) \\ h(\beta) = h(\gamma) \text{ and } H_1(\beta) = H_1(\gamma) \\ \qquad \text{and } H_2(\beta) > H_2(\gamma). \end{array}\right. \tag{18}$$

Pairwise combining conditions of (17) and (19), we obtain

$$\left[\begin{array}{l} h(\alpha) > h(\gamma) \\ h(\alpha) = h(\gamma) \text{ and } H_1(\alpha) > H_1(\gamma) \\ h(\alpha) = h(\gamma) \text{ and } H_1(\alpha) = H_1(\gamma) \\ \qquad \text{and } H_2(\alpha) > H_2(\gamma). \end{array}\right. \tag{19}$$

Then, $\alpha > \gamma$. □

Let $(\alpha_1, \ldots, \alpha_n)$ be a collection of PLNs, we denote:

$$\operatorname{arcmin}_h(\alpha_1, \ldots, \alpha_n) = \left\{\alpha_j \middle| h\left(\alpha_j\right) = \min\{h(\alpha_i)\}\right\},$$

$$\operatorname{arcmin}_{H_1}(\alpha_1, \ldots, \alpha_n) = \left\{\alpha_j \middle| H_1\left(\alpha_j\right) = \min\{H_1(\alpha_i)\}\right\},$$

$$\operatorname{arcmin}_{H_2}(\alpha_1, \ldots, \alpha_n) = \left\{\alpha_j \middle| H_2\left(\alpha_j\right) = \min\{H_2(\alpha_i)\}\right\},$$

$$\operatorname{arcmax}_h(\alpha_1, \ldots, \alpha_n) = \left\{\alpha_j \middle| h\left(\alpha_j\right) = \max\{h(\alpha_i)\}\right\},$$

$$\operatorname{arcmax}_{H_1}(\alpha_1, \ldots, \alpha_n) = \left\{\alpha_j \middle| H_1\left(\alpha_j\right) = \max\{H_1(\alpha_i)\}\right\},$$

$$\operatorname{arcmax}_{H_2}(\alpha_1, \ldots, \alpha_n) = \left\{\alpha_j \middle| H_2\left(\alpha_j\right) = \max\{H_2(\alpha_i)\}\right\}.$$

Definition 11. *Lower bound and upper bound of the collection of PLNs $(\alpha_1, \ldots, \alpha_n)$ are respectively defined as*

$$\alpha^- = \operatorname{arcmin}_{H_2}\left(\operatorname{arcmin}_{H_1}\left(\operatorname{arcmin}_h(\alpha_1, \ldots, \alpha_n)\right)\right),$$

$$\alpha^+ = \operatorname{arcmax}_{H_2}\left(\operatorname{arcmax}_{H_1}\left(\operatorname{arcmax}_h(\alpha_1, \ldots, \alpha_n)\right)\right).$$

Based on Definitions 9, 10 and 11, the following proposition can be easily proved.

Proposition 3. *For each collection of PLNs* $(\alpha_1,\ldots,\alpha_n)$,

$$\alpha^- \lesssim \alpha_i \lesssim \alpha^+, \forall i = 1,\ldots,n. \tag{20}$$

The $\lesssim$ *in the left of Eq. (20) means that for all* $\alpha_j \in \alpha^-$, *we have* $\alpha_j < \alpha_i$ *or* $\alpha_j \sim \alpha_i$. *Similar for the* $\lesssim$ *in the right.*

4. Aggregation operators of PLNs

In this section some operators, which aggregate PLNs, are proposed: picture linguistic arithmetic averaging (PLAA), picture linguistic weighted arithmetic averaging (PLWAA), picture linguistic ordered weighted averaging (PLOWA) and picture linguistic hybrid aggregation (PLHA) operators.

Throughout this paper, each weight vector is with respect to a collection of non-negative number with the total of 1.

Definition 12. *Picture linguistic arithmetic averaging (PLAA) operator is a mapping* $\mathrm{PLAA}: \Delta^n \to \Delta$ *defined as*

$$\mathrm{PLAA}(\alpha_1,\ldots,\alpha_n) = \frac{1}{n}(\alpha_1 \oplus \cdots \oplus \alpha_n), \tag{21}$$

where $(\alpha_1,\ldots,\alpha_n)$ *is a collection of PLNs.*

Definition 13. *Picture linguistic weighted arithmetic averaging (PLWAA) operator is a mapping* $\mathrm{PLWAA}: \Delta^n \to \Delta$ *defined as*

$$\mathrm{PLWAA}_w(\alpha_1,\ldots,\alpha_n) = w_1\alpha_1 \oplus \cdots \oplus w_n\alpha_n, \tag{22}$$

where $w = (w_1,\ldots,w_n)$ *is the weight vector of the collection of PLNs* $(\alpha_1,\ldots,\alpha_n)$.

Proposition 4. *Let* $(\alpha_1,\ldots,\alpha_n)$ *be a collection of PLNs, and* $w = (w_1,\ldots,w_n)$ *be the weight vector of this collection, then* $\mathrm{PLWAA}_w(\alpha_1,\ldots,\alpha_n)$ *is a PLN and*

$$\begin{aligned}&\mathrm{PLWAA}_w(\alpha_1,\ldots,\alpha_n) = \\ &\left\langle s_{\sum_{i=1}^{n} w_i\theta(\alpha_i)}, \frac{\sum_{i=1}^{n} w_i\theta(\alpha_i)\mu(\alpha_i)}{\sum_{i=1}^{n} w_i\theta(\alpha_i)}, \right. \\ &\left. \frac{\sum_{i=1}^{n} w_i\theta(\alpha_i)\eta(\alpha_i)}{\sum_{i=1}^{n} w_i\theta(\alpha_i)}, \frac{\sum_{i=1}^{n} w_i\theta(\alpha_i)\nu(\alpha_i)}{\sum_{i=1}^{n} w_i\theta(\alpha_i)} \right\rangle.\end{aligned} \tag{23}$$

Proof. By Definition 7, aggregated value by using PLWAA is also a PLN. In the next step, we prove (23) by using mathematical induction on n.
1) For $n = 2$: By Definition 7,

$$w_1\alpha_1 = \langle s_{w_1\theta(\alpha_1)}, \mu(\alpha_1), \eta(\alpha_1), \nu(\alpha_1)\rangle, \tag{24}$$

and

$$w_2\alpha_2 = \langle s_{w_2\theta(\alpha_2)}, \mu(\alpha_2), \eta(\alpha_2), \nu(\alpha_2)\rangle. \tag{25}$$

We thus obtain

$$\begin{aligned}&w_1\alpha_1 \oplus w_2\alpha_2 = \langle s_{w_1\theta(\alpha_1)+w_2\theta(\alpha_2)}, \\ &\frac{w_1\theta(\alpha_1)\mu(\alpha_1) + w_2\theta(\alpha_2)\mu(\alpha_2)}{w_1\theta(\alpha_1) + w_2\theta(\alpha_2)}, \\ &\frac{w_1\theta(\alpha_1)\eta(\alpha_1) + w_2\theta(\alpha_2)\eta(\alpha_2)}{w_1\theta(\alpha_1) + w_2\theta(\alpha_2)}, \\ &\left.\frac{w_1\theta(\alpha_1)\nu(\alpha_1) + w_2\theta(\alpha_2)\nu(\alpha_2)}{w_1\theta(\alpha_1) + w_2\theta(\alpha_2)}\right\rangle,\end{aligned} \tag{26}$$

i. e., (23) holds for $n = 2$.
2) Let us assume that (23) holds for $n = k$ ($k \geq 2$), that is

$$\begin{aligned}&w_1\alpha_1 \oplus \ldots \oplus w_k\alpha_k = \\ &\left\langle s_{\sum_{i=1}^{k} w_i\theta(\alpha_i)}, \frac{\sum_{i=1}^{k} w_i\theta(\alpha_i)\mu(\alpha_i)}{\sum_{i=1}^{k} w_i\theta(\alpha_i)}, \right. \\ &\left. \frac{\sum_{i=1}^{k} w_i\theta(\alpha_i)\eta(\alpha_i)}{\sum_{i=1}^{k} w_i\theta(\alpha_i)}, \frac{\sum_{i=1}^{k} w_i\theta(\alpha_i)\nu(\alpha_i)}{\sum_{i=1}^{k} w_i\theta(\alpha_i)} \right\rangle.\end{aligned} \tag{27}$$

Then,

$$\begin{aligned}&w_1\alpha_1 \oplus \ldots \oplus w_k\alpha_k \oplus w_{k+1}\alpha_{k+1} \\ &= \left\langle s_{\sum_{i=1}^{k} w_i\theta(\alpha_i)}, \frac{\sum_{i=1}^{k} w_i\theta(\alpha_i)\mu(\alpha_i)}{\sum_{i=1}^{k} w_i\theta(\alpha_i)}, \right. \\ &\left. \frac{\sum_{i=1}^{k} w_i\theta(\alpha_i)\eta(\alpha_i)}{\sum_{i=1}^{k} w_i\theta(\alpha_i)}, \frac{\sum_{i=1}^{k} w_i\theta(\alpha_i)\nu(\alpha_i)}{\sum_{i=1}^{k} w_i\theta(\alpha_i)} \right\rangle \oplus \\ &\langle s_{w_{k+1}\theta(\alpha_{k+1})}, \mu(\alpha_{k+1}), \eta(\alpha_{k+1}), \nu(\alpha_{k+1})\rangle\end{aligned}$$

$$=\left\langle s_{\left(\sum_{i=1}^{k} w_i\theta(\alpha_i)\right)+w_{k+1}\alpha_{k+1}}, \frac{\left(\sum_{i=1}^{k} w_i\theta(\alpha_i)\mu(\alpha_i)\right)+w_{k+1}\theta(\alpha_{k+1})\mu(\alpha_{k+1})}{\left(\sum_{i=1}^{k} w_i\theta(\alpha_i)\right)+w_{k+1}\theta(\alpha_{k+1})}, \frac{\left(\sum_{i=1}^{k} w_i\theta(\alpha_i)\eta(\alpha_i)\right)+w_{k+1}\theta(\alpha_{k+1})\eta(\alpha_{k+1})}{\left(\sum_{i=1}^{k} w_i\theta(\alpha_i)\right)+w_{k+1}\theta(\alpha_{k+1})}, \frac{\left(\sum_{i=1}^{k} w_i\theta(\alpha_i)\nu(\alpha_i)\right)+w_{k+1}\theta(\alpha_{k+1})\nu(\alpha_{k+1})}{\left(\sum_{i=1}^{k} w_i\theta(\alpha_i)\right)+w_{k+1}\theta(\alpha_{k+1})}\right\rangle$$

$$=\left\langle s_{k+1 \atop \sum_{i=1} w_i\theta(\alpha_i)}, \frac{\sum_{i=1}^{k+1} w_i\theta(\alpha_i)\mu(\alpha_i)}{\sum_{i=1}^{k+1} w_i\theta(\alpha_i)}, \frac{\sum_{i=1}^{k+1} w_i\theta(\alpha_i)\eta(\alpha_i)}{\sum_{i=1}^{k+1} w_i\theta(\alpha_i)}, \frac{\sum_{i=1}^{k+1} w_i\theta(\alpha_i)\nu(\alpha_i)}{\sum_{i=1}^{k+1} w_i\theta(\alpha_i)}\right\rangle.$$

This implies that, (23) holds for $n = k + 1$, which completes the proof. □

According to Definitions 9, 10, 13, Propositions 3 and 4, it can be easily proved that the PLWAA operator has the following properties. Let $(\alpha_1, \dots, \alpha_n)$ be a collection of PLNs with the weight vector $w = (w_1, \dots, w_n)$, we have:

(1) Idempotency: If $\alpha_i = \alpha$ for all $i = 1, \dots, n$,

$$\text{PLWAA}_w(\alpha_1, \dots, \alpha_n) = \alpha.$$

(2) Boundary:

$$\alpha^- \lesssim \text{PLWAA}_w(\alpha_1, \dots, \alpha_n) \lesssim \alpha^+.$$

(3) Monotonicity: Let $\left(\alpha_1^*, \dots, \alpha_n^*\right)$ be a collection of PLNs such that $\alpha_i^* \le \alpha_i$ for all $i = 1, \dots, n$, then

$$\text{PLWAA}_w(\alpha_1^*, \dots, \alpha_n^*) \le \text{PLWAA}_w(\alpha_1, \dots, \alpha_n).$$

(4) Commutativity:

$$\text{PLWAA}_w(\alpha_1, \dots, \alpha_n) = \text{PLWAA}_{w'}\left(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}\right),$$

where σ is any permutation on the set $\{1, \dots, n\}$ and $w' = (w_{\sigma(1)}, \dots, w_{\sigma(n)})$.

(5) Associativity: Consider an added collection of PLNs $(\gamma_1, \dots, \gamma_m)$ with the associated weight vector $w' = \left(w'_1, \dots, w'_m\right)$,

$$\begin{aligned}&\text{PLWAA}_u(\alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_m)\\ =&\text{PLWAA}_v(\text{PLWAA}_w(\alpha_1, \dots, \alpha_n),\\ &\quad \text{PLWAA}_{w'}(\gamma_1, \dots, \gamma_m)),\end{aligned}$$

where $u = \left(\frac{w_1}{2}, \dots, \frac{w_n}{2}, \frac{w'_1}{2}, \dots, \frac{w'_m}{2}\right)$ and $v = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Definition 14. *Picture linguistic ordered weighted averaging (PLOWA) operator is a mapping* PLOWA : $\Delta^n \to \Delta$ *defined as*

$$\text{PLOWA}_\omega(\alpha_1, \dots, \alpha_n) = \omega_1\beta_1 \oplus \dots \oplus \omega_n\beta_n, \quad (28)$$

where $\omega = (\omega_1, \dots, \omega_n)$ *is the weight vector of the* PLOWA *operator and* $\beta_j \in \Delta$ $(j = 1, \dots, n)$ *is the j-th largest of the totally comparable collection of PLNs* $(\alpha_1, \dots, \alpha_n)$.

Definition 14 requires that all pairs of PLNs of the collection $(\alpha_1, \dots, \alpha_n)$ are comparable. We further consider the cases when the collection $(\alpha_1, \dots, \alpha_n)$ is not totally comparable. If $\alpha_i \sim \alpha_j$ and $\theta(\alpha_i) < \theta\left(\alpha_j\right)$, we assign α_j to α_i. It is reasonable since α_i and α_j have the same score, first accuracy and second accuracy.

Example 3. *Let us consider* $\alpha_1 = \langle s_2, 0.2, 0.4, 0.4\rangle$, $\alpha_2 = \langle s_4, 0.2, 0.3, 0.3\rangle$, $\alpha_3 = \langle s_2, 0.1, 0.2, 0.6\rangle$, $\alpha_4 = \langle s_4, 0.1, 0.2, 0.2\rangle$ *and* $\omega = (0.2, 0.4, 0.15, 0.25)$. *Taking Definitions 9 and 10 into account, we get*

$$\alpha_2 > \alpha_1 \sim \alpha_4 > \alpha_3. \quad (29)$$

α_4 *is assigned to* α_1. *By adding the* 2*-th and* 3*-th position of weight vector* ω, *we obtain* $\omega' = (0.2, 0.55, 0.25)$. *Hence,*

$$\text{PLOWA}_\omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \text{PLOWA}_{\omega'}(\alpha_1, \alpha_2, \alpha_3).$$

In this case, $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1$ *and* $\beta_3 = \alpha_3$.

In the same way as in Proposition 4, we have the following proposition.

Proposition 5. *Let* $(\alpha_1, \dots, \alpha_n)$ *be a collection of PLNs, and* $\omega = (\omega_1, \dots, \omega_n)$ *be the weight vector of* *the* PLOWA*, then* $\mathrm{PLOWA}_{\omega}(\alpha_1,\ldots,\alpha_n)$ *is a PLN and*

$$\mathrm{PLOWA}_{\omega}(\alpha_1,\ldots,\alpha_n)=\left\langle s_{\sum_{j=1}^{n}\omega_j\theta(\beta_j)}, \frac{\sum_{j=1}^{n}\omega_j\theta(\beta_j)\mu(\beta_j)}{\sum_{j=1}^{n}\omega_j\theta(\beta_j)}, \frac{\sum_{j=1}^{n}\omega_j\theta(\beta_j)\eta(\beta_j)}{\sum_{j=1}^{n}\omega_j\theta(\beta_j)}, \frac{\sum_{j=1}^{n}\omega_j\theta(\beta_j)\nu(\beta_j)}{\sum_{j=1}^{n}\omega_j\theta(\beta_j)}\right\rangle, \quad (30)$$

with β_j $(j=1,\ldots,n)$ *is the j-th largest of the collection* $(\alpha_1,\ldots,\alpha_n)$.

Example 4. *(Continuation of Example 3) We have*

$$\mathrm{PLOWA}_{\omega'}(\alpha_1,\alpha_2,\alpha_3)=\bar{\alpha}, \quad (31)$$

where $\bar{\alpha}$ *is determined as follows.*

$$\theta(\bar{\alpha})=\omega'_1\times\theta(\beta_1)+w'_2\times\theta(\beta_2)+w'_3\times\theta(\beta_3) = 0.2\times 4+0.55\times 2+0.25\times 2=2.4,$$

$$\begin{aligned}&\mu(\bar{\alpha})\\ =&\big(w'_1\times\theta(\beta_1)\times\mu(\beta_1)+w'_2\times\theta(\beta_2)\times\mu(\beta_2)\\ &+w'_3\times\theta(\beta_3)\times\mu(\beta_3)\big)/\theta(\bar{\alpha})\\ =&\frac{0.2\times 4\times 0.2+0.55\times 2\times 0.2+0.25\times 2\times 0.2}{2.4}\\ =&0.2.\end{aligned}$$

As a similarity, $\eta(\bar{\alpha})=0.325$ *and* $\nu(\bar{\alpha})=0.408$*. We finally get*

$$\mathrm{PLOWA}_{\omega}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\langle s_{2.4},0.2,0.325,0.408\rangle.$$

The PLOWA can be shown to satisfy the properties of idempotency, boundary, monotonicity, commutativity and associativity. Let $(\alpha_1,\ldots,\alpha_n)$ be a totally comparable collection of PLNs, and $\omega=(\omega_1,\ldots,\omega_n)$ be the weight vector of the PLOWA operator, then

(1) Idempotency: If $\alpha_i=\alpha$ for all $i=1,\ldots,n$, then

$$\mathrm{PLOWA}_{\omega}(\alpha_1,\ldots,\alpha_n)=\alpha;$$

(2) Boundary:

$$\min_{i=1,\ldots,n}\{\alpha_i\}\le\mathrm{PLOWA}_{\omega}(\alpha_1,\ldots,\alpha_n)\le\max_{i=1,\ldots,n}\{\alpha_i\};$$

(3) Monotonicity: Let $\left(\alpha_1^*,\ldots,\alpha_n^*\right)$ be a totally comparable collection of PLNs such that $\alpha_i^*\le\alpha_i$ for all $i=1,\ldots,n$, then

$$\mathrm{PLOWA}_{\omega}(\alpha_1^*,\ldots,\alpha_n^*)\le\mathrm{PLOWA}_{\omega}(\alpha_1,\ldots,\alpha_n);$$

(4) Commutativity:

$$\mathrm{PLOWA}_{\omega}(\alpha_1,\ldots,\alpha_n)=\mathrm{PLOWA}_{\omega}(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(n)}),$$

where σ is any permutation on the set $\{1,\ldots,n\}$.

(5) Associativity: Consider an added totally comparable collection of PLNs $(\gamma_1,\ldots,\gamma_m)$ with the associated weight vector $\omega'=\left(\omega'_1,\ldots,\omega'_m\right)$. If $\alpha_1\ge\ldots\ge\alpha_n\ge\gamma_1\ge\ldots\ge\gamma_m$,

$$\begin{aligned}&\mathrm{PLOWA}_{\epsilon}(\alpha_1,\ldots,\alpha_n,\gamma_1,\ldots,\gamma_m)\\ =&\mathrm{PLOWA}_{\delta}(\mathrm{PLOWA}_{\omega}(\alpha_1,\ldots,\alpha_n),\\ &\mathrm{PLOWA}_{\omega'}(\gamma_1,\ldots,\gamma_m)),\end{aligned}$$

where $\epsilon=\left(\frac{\omega_1}{2},\ldots,\frac{\omega_n}{2},\frac{\omega'_1}{2},\ldots,\frac{\omega'_m}{2}\right)$ and $\delta=\left(\frac{1}{2},\frac{1}{2}\right)$.

Proposition 6 shows some special cases of the PLOWA operator.

Proposition 6. *Let* $(\alpha_1,\ldots,\alpha_n)$ *be a totally comparable collection of PLNs, and* $\omega=(\omega_1,\ldots,\omega_n)$ *be the weight vector, then*

(1) If $\omega=(1,0,\ldots,0)$*, then* $\mathrm{PLOWA}_{\omega}(\alpha_1,\ldots,\alpha_n)=\max_{i=1,\ldots,n}\{\alpha_i\}$;

(2) If $\omega=(0,\ldots,0,1)$*, then* $\mathrm{PLOWA}_{\omega}(\alpha_1,\ldots,\alpha_n)=\min_{i=1,\ldots,n}\{\alpha_i\}$;

(3) If $\omega_j=1$*, and* $\omega_i=0$ *for all* $i\ne j$*, then* $\mathrm{PLOWA}_{\omega}(\alpha_1,\ldots,\alpha_n)=\beta_j$ *where* β_j *is the j-th largest of the collection of PLNs* $(\alpha_1,\ldots,\alpha_n)$.

Definition 15. *Picture Linguistic hybrid averaging (PLHA) operator for PLNs is a mapping* $\mathrm{PLHA}:\Delta^n\to\Delta$ *defined as*

$$\mathrm{PLHA}_{w,\omega}(\alpha_1,\ldots,\alpha_n)=\omega_1\beta'_1\oplus\cdots\oplus\omega_n\beta'_n;$$

where ω *is the associated weight vector of the PLHA operator, and* β'_j *is the j-largest of the totally comparable collection of ILNs* $(nw_1\alpha_1,\ldots,nw_n\alpha_n)$ *with* $w=(w_1,\ldots,w_n)$ *is the weight vector of the collection of PLNs* $(\alpha_1,\ldots,\alpha_n)$.

The Proposition 7 gives the explicit formula for PLHA operator.

Proposition 7. *Let $(\alpha_1, \ldots, \alpha_n)$ be a collection of PLNs, $\omega = (\omega_1, \ldots, \omega_n)$ be the associated vector of the PLHA operator, and $w = (w_1, \ldots, w_n)$ be the weight vector of $(\alpha_1, \ldots, \alpha_n)$, then $PLHA_{w,\omega}(\alpha_1, \ldots, \alpha_n)$ is a PLNs and*

$$\mathrm{PLHA}_{w,\omega}(\alpha_1, \ldots, \alpha_n) = \left\langle s_{\sum_{j=1}^{n} \omega_j \theta(\beta'_j)}, \frac{\sum_{j=1}^{n} \omega_j \theta(\beta'_j) \mu(\beta'_j)}{\sum_{j=1}^{n} \omega_j \theta(\beta'_j)}, \frac{\sum_{j=1}^{n} \omega_j \theta(\beta'_j) \eta(\beta'_j)}{\sum_{j=1}^{n} \omega_j \theta(\beta'_j)}, \frac{\sum_{j=1}^{n} \omega_j \theta(\beta'_j) \nu(\beta_j)}{\sum_{j=1}^{n} \omega_j \theta(\beta'_j)} \right\rangle, \quad (32)$$

where β'_j is the j-largest of the totally comparable collection of ILNs $(nw_1\alpha_1, \ldots, nw_n\alpha_n)$.

Similar to PLWAA and PLOWA operators, the PLHA operator is idempotent, bounded, monotonous, commutative and associative. Let $(\alpha_1, \ldots, \alpha_n)$ be a collection of PLNs, $\omega = (\omega_1, \ldots, \omega_n)$ be the associated vector of the PLHA operator, and $w = (w_1, \ldots, w_n)$ be the weight vector of $(\alpha_1, \ldots, \alpha_n)$, then

(1) Idempotency: If $\alpha_i = \alpha$ for all $i = 1, \ldots, n$, then

$$\mathrm{PLHA}_{w,\omega}(\alpha_1, \ldots, \alpha_n) = \alpha;$$

(2) Boundary:

$$\alpha^- \lesssim \mathrm{PLHA}_{w,\omega}(\alpha_1, \ldots, \alpha_n) \lesssim \alpha^+;$$

(3) Monotonicity: Let $(\alpha_1^*, \ldots, \alpha_n^*)$ be a collection of PLNs such that $\alpha_i^* \lesssim \alpha_i$ for all $i = 1, \ldots, n$, then

$$\mathrm{PLHA}_{w,\omega}(\alpha_1^*, \ldots, \alpha_n^*) \lesssim \mathrm{PLHA}_{w,\omega}(\alpha_1, \ldots, \alpha_n);$$

(4) Commutativity:

$$\mathrm{PLHA}_{w,\omega}(\alpha_1, \ldots, \alpha_n) = \mathrm{PLHA}_{w,\omega}(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}),$$

where σ is any permutation on the set $\{1, \ldots, n\}$ and $w' = (w_{\sigma(1)}, \ldots, w_{\sigma(n)})$.

(5) Associativity: Consider an added collection of PLNs $(\gamma_1, \ldots, \gamma_m)$ with the associated weight vector $w' = (w'_1, \ldots, w'_m)$ such that $nw_1\alpha_1 \geq \cdots \geq nw_n\alpha_n \geq mw'_1\gamma_1 \geq \cdots \geq mw'_m\gamma_m$. We have

$$\begin{aligned} &\mathrm{PLHA}_{u,\epsilon}(\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m) \\ =&\mathrm{PLHA}_{v,\delta}(\mathrm{PLHA}_{w,\omega}(\alpha_1, \ldots, \alpha_n), \\ &\mathrm{PLHA}_{w',\omega'}(\gamma_1, \ldots, \gamma_m)), \end{aligned}$$

where $u = \left(\frac{w_1}{2}, \ldots, \frac{w_n}{2}, \frac{w'_1}{2}, \ldots, \frac{w'_m}{2}\right)$, $\epsilon = \left(\frac{\omega_1}{2}, \ldots, \frac{\omega_n}{2}, \frac{\omega'_1}{2}, \ldots, \frac{\omega'_m}{2}\right)$ and $v = \delta = \left(\frac{1}{2}, \frac{1}{2}\right)$.

We can prove that the PLWAA and PLOWA operators are two special cases of the PLHA operator as in Proposition 8.

Proposition 8. *If $\omega = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$, the PLHA operator is reduced to the PLWAA operator; and if $w = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$, the PLHA operator is reduced to the PLOWA operator.*

5. GDM under picture linguistic assessments

Let us consider a hypothetical situation, in which $A = \{A_1, \ldots, A_m\}$ is the set of alternatives, and $C = \{C_1, \ldots, C_n\}$ is the set of criteria with the weight vector $c = (c_1, \ldots, c_n)$. We assume that $D = \{d_1, \ldots, d_p\}$ is a set of decision makers (DMs), and $w = (w_1, \ldots, w_p)$ is the weight vector of DMs. Each DM d_k presents the characteristic of the alternative A_i with respect to the criteria C_j by the PLN $\alpha_{ij}^{(k)} = \left\langle s_{\theta(\alpha_{ij}^{(k)})}, \mu_{\alpha_{ij}^{(k)}}, \eta_{\alpha_{ij}^{(k)}}, \nu_{\alpha_{ij}^{(k)}} \right\rangle$ $(i = 1, \ldots, m,\ j = 1, \ldots, n,\ k = 1, \ldots, p)$. The decision matrix R_k is given by $R_k = \left(\alpha_{ij}^{(k)}\right)_{m \times n}$ $(k = 1, \ldots, p)$. The alternatives will be ranked by the following algorithm.

Step 1. Derive the overall values $\alpha_i^{(k)}$ of the alternatives A_i, given by the DM d_k:

$$\alpha_i^{(k)} = \mathrm{PLWAA}_c\left(\alpha_{i1}^{(k)}, \ldots, \alpha_{in}^{(k)}\right), \quad (33)$$

for $i = 1, \ldots, m$, and $k = 1, \ldots, p$.

Table 1. Decision matrix R_1

	C_1	C_2	C_3
A_1	$\langle s_4, 0.6, 0.1, 0.2\rangle$	$\langle s_4, 0.4, 0.2, 0.2\rangle$	$\langle s_5, 0.2, 0.3, 0.5\rangle$
A_2	$\langle s_5, 0.7, 0.2, 0.1\rangle$	$\langle s_4, 0.4, 0.1, 0.4\rangle$	$\langle s_4, 0.5, 0.2, 0.3\rangle$
A_3	$\langle s_5, 0.3, 0.1, 0.4\rangle$	$\langle s_5, 0.4, 0.3, 0.3\rangle$	$\langle s_6, 0.7, 0.1, 0.2\rangle$
A_4	$\langle s_4, 0.6, 0.1, 0.2\rangle$	$\langle s_4, 0.6, 0.1, 0.2\rangle$	$\langle s_5, 0.3, 0.1, 0.5\rangle$

Step 2. Derive the collective overall values α_i by aggregating the individual overall values $\alpha_i^{(1)}, \ldots, \alpha_i^{(p)}$:

$$\alpha_i = \text{PLHA}_{w,\omega}\left(\alpha_i^{(1)}, \ldots, \alpha_i^{(p)}\right), \qquad (34)$$

where $\omega = \left(\omega_1, \ldots, \omega_p\right)$ is the weight vector of the PLHA operator ($i = 1, \ldots, m$).

Step 3. Calculate the scores $h(\alpha_i)$, first accuracies $H_1(\alpha_i)$ and second accuracies $H_2(\alpha_i)$ ($i = 1, \ldots, m$), rank the alternatives by using Definition 9 (the alternative A_{i_1} is called to be better than the alternative A_{i_2}, denoted by $A_{i_1} > A_{i_2}$, iff $\alpha_{i_1} > \alpha_{i_2}$, for all $i_1, i_2 = 1, \ldots, m$).

6. An illutrative example

This situation concerns four alternative enterprises, which will be chosen by three DMs whose weight vector is $w = (0.3, 0.4, 0.3)$. The enterprises will be considered under three criteria C_1, C_2 and C_3. Assume that the weight vector of the criteria is $c = (0.37, 0.35, 0.28)$. Three decision matrices are listed in Tabs. 1, 2 and 3.

Step 1. Using explicit form of the PLWAA operation given in Eq. 23, we obtain overall values $\alpha_i^{(k)}$ of the alternatives A_i given by the DMs d_k ($i = 1, 2, 3, 4$ and $k = 1, 2, 3$) as in Tab. 4.

Step 2. Aggregate all the individual overall values $\alpha_i^{(1)}$, $\alpha_i^{(2)}$ and $\alpha_i^{(3)}$ of the alternatives A_i ($i = 1, 2, 3, 4$) by the PLHA operator with associated weight vector $\omega = (0.2, 0.5, 0.3)$.

$$\begin{aligned}
\alpha_1 &= \langle s_{4.40}, 0.3965, 0.2045, 0.3438\rangle, \\
\alpha_2 &= \langle s_{4.57}, 0.3481, 0.1428, 0.4040\rangle, \\
\alpha_3 &= \langle s_{5.32}, 0.3628, 0.1666, 0.4050\rangle, \\
\alpha_4 &= \langle s_{5.16}, 0.4098, 0.1510, 0.3948\rangle.
\end{aligned}$$

Step 3. By eq. (12),

$$\begin{aligned}
h(\alpha_1) = 0.2318, h(\alpha_2) = -0.2556 \\
h(\alpha_3) = -0.2246, h(\alpha_4) = 0.078.
\end{aligned}$$

By Definition 9,

$$h(\alpha_1) > h(\alpha_4) > h(\alpha_3) > h(\alpha_2)$$

then $A_1 > A_4 > A_3 > A_2$.

7. Conclusion

In this paper, motivated by picture fuzzy sets and linguistic approaches, the notion of picture linguistic numbers are first defined. We propose the score, first accuracy and second accuracy of picture linguistic numbers, and propose a simple approach for the comparison between two picture linguistic numbers. Simultaneously, the operation laws for picture linguistic numbers are given and the accompanied properties are studied. Further, some aggregation operators are developed: picture linguistic arithmetic averaging, picture linguistic weighted

Table 2. Decision matrix R_2

	C_1	C_2	C_3
A_1	$\langle s_4, 0.7, 0.1, 0.2\rangle$	$\langle s_6, 0.2, 0.2, 0.5\rangle$	$\langle s_4, 0.7, 0.2, 0.1\rangle$
A_2	$\langle s_3, 0.2, 0.2, 0.6\rangle$	$\langle s_5, 0.5, 0.1, 0.2\rangle$	$\langle s_5, 0.3, 0.1, 0.4\rangle$
A_3	$\langle s_4, 0.2, 0.1, 0.5\rangle$	$\langle s_7, 0.2, 0.2, 0.6\rangle$	$\langle s_5, 0.1, 0.2, 0.6\rangle$
A_4	$\langle s_5, 0.7, 0.2, 0.1\rangle$	$\langle s_5, 0.2, 0.1, 0.7\rangle$	$\langle s_4, 0.6, 0.1, 0.2\rangle$

Table 3. Decision matrix R_3

	C_1	C_2	C_3
A_1	$\langle s_4, 0.6, 0.3, 0.1\rangle$	$\langle s_6, 0.2, 0.3, 0.5\rangle$	$\langle s_5, 0.2, 0.1, 0.7\rangle$
A_2	$\langle s_3, 0.2, 0.2, 0.5\rangle$	$\langle s_5, 0.2, 0.1, 0.6\rangle$	$\langle s_6, 0.2, 0.2, 0.6\rangle$
A_3	$\langle s_5, 0.3, 0.2, 0.5\rangle$	$\langle s_7, 0.8, 0.1, 0.1\rangle$	$\langle s_5, 0.2, 0.2, 0.5\rangle$
A_4	$\langle s_3, 0.7, 0.1, 0.2\rangle$	$\langle s_5, 0.2, 0.2, 0.5\rangle$	$\langle s_6, 0.3, 0.1, 0.6\rangle$

Table 4. Overall values $\alpha_i^{(k)}$ of the alternatives A_i given by the DMs d_k ($i = 1, 2, 3, 4$; $k = 1, 2, 3$)

	d_1	d_2	d_3
A_1	$\langle s_{4.28}, 0.4037, 0.1981, 0.2981\rangle$	$\langle s_{4.70}, 0.4766, 0.1685, 0.3102\rangle$	$\langle s_{4.98}, 0.3189, 0.2438, 0.4373\rangle$
A_2	$\langle s_{4.37}, 0.5526, 0.1680, 0.2474\rangle$	$\langle s_{4.26}, 0.3561, 0.1261, 0.3700\rangle$	$\langle s_{4.54}, 0.2000, 0.1615, 0.5756\rangle$
A_3	$\langle s_{5.28}, 0.4604, 0.1663, 0.3032\rangle$	$\langle s_{5.33}, 0.1737, 0.1722, 0.5722\rangle$	$\langle s_{5.70}, 0.4904, 0.1570, 0.3281\rangle$
A_4	$\langle s_{4.28}, 0.5019, 0.1000, 0.2981\rangle$	$\langle s_{5.28}, 0.4070, 0.1682, 0.3917\rangle$	$\langle s_{4.54}, 0.3593, 0.1385, 0.4637\rangle$

arithmetic averaging, picture linguistic ordered weighted averaging and picture linguistic hybrid aggregation operators. Finally, based on the picture linguistic weighted arithmetic averaging and the picture linguistic hybrid aggregation operators, we propose an approach to handle multi-criteria group decision making problems under picture linguistic environment.

Acknowledgments

This research is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 102.01- 2017.02.

References

[1] G. I. Adamopoulos and G. P. Pappis, A fuzzy linguistic approach to a multicriteria sequencing problem, *European Journal of Operational Research* **92** (1996) 628–636.

[2] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20** (1986) 87–96.

[3] K. T. Atanassov and S. Stoeva, Intuitionistic L-fuzzy sets, in *Cybernetics and Systems Research*, eds. R. Trappl (Elsevier Science Pub., Amsterdam, 1986), pp. 539–540.

[4] P. Chang and Y. Chen, A fuzzy multicriteria decision making method for technology transfer strategy selection in biotechnology, *Fuzzy Sets and Systems* **63** (1994) 131-139.

[5] S. M. Chen, A new method for tool steel materials selection under fuzzy environment, *Fuzzy Sets and Systems* **92** (1997) 265–274.

[6] Z. C. Chen, P. H. Liu and Z. Pei, An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers, *International Journal of*

Computational Intelligence Systems **8** (4) (2015) 747–760.

[7] B. C. Cuong and V. Kreinovich, Picture fuzzy sets - a new concept for computational intelligence problems, in *Proc. 3rd world congress on information and communication technologies (WICT 2013)* pp. 1–6.

[8] B. C. Cuong and P. H. Phong, Max - Min Composition of Linguistic Intuitionistic Fuzzy Relations and Application in Medical Diagnosis, *VNU Journal of Science: Comp. Science & Com. Eng.* **30** (4) (2014) 601–968.

[9] F. Herrera and J. L. Verdegay, *Linguistic assessments in group decision*", in *Proc. First European Congress on Fuzzy and Intelligent Technologies* (Aachen, 1993) pp. 941–948.

[10] F. Herrera and E. Herrera-Viedma, Aggregation operators for linguistic weighted information, *IEEE Transactions on Systems, Man, and Cybernetics-Part A* **27** (1997), 646–656.

[11] F. Herrera and L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, *IEEE Transactions on Fuzzy Systems* **8** (2000) 746–752.

[12] F. Herrera and E. Herrera-Viedma, Choice functions and mechanisms for linguistic preference relations, *European Journal of Operational Research* **120** (2000) 144–161.

[13] G. J. Klir and B. Yuan, Fuzzy sets an fuzzy logic: Theory and Applications (Prentice-Hall PTR, 1995).

[14] C. K. Law, Using fuzzy numbers in educational grading systems, *Fuzzy Sets and Systems* **83** (1996) 311–323.

[15] H. M. Lee, Applying fuzzy set theory to evaluate the rate of aggregative risk in software development, *Fuzzy Sets and Systems* **80** (1996) 323–336.

[16] P. H. Phong and B. C. Cuong, Some intuitionistic linguistic aggregation operators, *Journal of Computer Science and Cybernetics* **30** (3) (2014) 216–226.

[17] P. H. Phong and B. C. Cuong, Symbolic computational models for intuitionistic linguistic information, *Journal of Computer Science and Cybernetics* **32** (1) (2016) 30–44.

[18] P. Singh, Correlation coefficients for picture fuzzy sets, *Journal of Intelligent & Fuzzy Systems***28** (2) (2015) 591-604.

[19] L. H. Son, DPFCM: A novel distributed picture fuzzy clustering method on picture fuzzy sets, *Expert Systems with Applications* **42** (2015) 51–66.

[20] P. H. Thong and L. H. Son, Picture fuzzy clustering: a new computational intelligence method, *Soft Computing* DOI 10.1007/s00500-015-1712-7

[21] V. Torra, The weighted OWA operator, *International Journal of Intelligent Systems* **12** (1997) 153–166.

[22] J. Q. Wang and H. B. Li, Multi-criteria decision-making method based on aggregation operators for intuitionistic linguistic fuzzy numbers, *Control and Decision* **25** (10) (2010) 1571–1574, 1584.

[23] X. F. Wang, J. Q. Wang and W. E. Yang, Multi-criteria group decision making method based on intuitionistic linguistic aggregation operators, *Journal of Intelligent & Fuzzy Systems* **26** (2014), 115–125.

[24] Z. S. Xu, Uncertain Multiple Attribute Decision Making: Methods and Applications (Tsinghua University Press, Beijing, 2004).

[25] Z. S. Xu , Method based on fuzzy linguistic assessments and GIOWA operator in multi-attribute group decision making, *Journal of Systems Science and Mathematical Sciences* **24** (2004) 218–224.

[26] Z. S. Xu, On generalized induced linguistic aggregation operators, *International Journal of General Systems* **35** (2006) 17–28.

[27] Z. S. Xu, A note on linguistic hybrid arithmetic averaging operator in group decision making with linguistic information, *Group Decision and Negotiation* **15** (2006) 581–591.

[28] R. R. Yager, A new methodology for ordinal multiobjective decisions based on fuzzy sets, *Decision Sciences* **12** (1981) 589–600.

[29] R. R. Yager, Applications and extensions of OWA aggregations, *International Journal of Man-Machine Studied* **37** (1992) 103–132.

[30] R. R. Yager and A. Rybalov, Understanding the median as a fusion operator, *International Journal of General Systems* **26** (1997) 239-263.

[31] L. A. Zadeh, Fuzzy sets, *Information and Control* **8**, 338–353.
[32] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, *Information Sciences* **8** (3) (1975)199–249.